

# Hadronic decay width from finite-volume energy spectrum in lattice QCD

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**Abstract.** The standard approach to determine the parameters of a resonance is based on the study of the volume dependence of the energy spectrum. In this work we study a non-linear sigma model coupled to a scalar field in which a resonance emerges. Using an analysis method introduced recently, based on the concept of probability distribution, it is possible to determine the mass and the width of the resonance.

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## INTRODUCTION

Lattice simulations provide a non-perturbative framework for calculating many low energy properties of *QCD*. Of these studies, the mass spectrum of the lightest hadrons is amongst the most important. In a lattice calculation the (stable) hadron spectrum is determined by numerical measurement of the energy eigenvalues of the *QCD* Hamiltonian. Nonetheless, most hadronic states are resonances which cannot be identified with a single energy level of the Hamiltonian: this complicates their mass and the width determination. A standard approach, introduced by Lüscher [1, 2, 3] (see also [4, 5] and for a generalization in moving frames [6]), to extract resonance parameters, starts by studying the volume dependence of the energy spectrum of the Hamiltonian when the system is confined in a finite box. There is a connection between the two-particle energy spectrum in finite volume and the elastic scattering phase shift in the infinite volume.

It was noted that where the phase shift in infinite volume quickly passes through  $\pi/2$  an abrupt rearrangement of the finite volume energy levels occurs. This peculiar behaviour, known as an avoided level crossing (ALC), is the signature of a resonance in the lattice data. Therefore, it is possible to extract the phase shift from lattice simulations and, consequently, to determine the position (the mass value) and the width of the resonances. The feasibility of this method has been demonstrated in four dimensional Ising model [7] and for models in two dimensions: in particular in the  $O(3)$  non-linear  $\sigma$ -model [8] and in two coupled Ising models [9]. Subsequently, it was applied in three dimensions in *QED* to study the meson-meson scattering phase shift [10] and in four dimensional  $O(4)$   $\phi^4$  theory [11]. Recently the method has been used to extract the  $\rho$  resonance from lattice *QCD* [12, 13] and to calculate the S-wave pion-pion scattering length in the isospin  $I = 2$  channel and the P-wave pion-pion scattering phase in the isospin  $I = 1$  channel [14].

During the last years many steps have been taken to address the problem of resonances on lattice; however, the approach described before is limited. Many of the resonances in *QCD* are characterized by a large width and so the rearrangement of the finite-volume energy levels does not provide a clear signature of the presence of a resonance: the ALC is washed out. A possible solution to this problem comes from a new method to analyze the finite volume energy spectrum [15]. The idea behind this work is to test this method in the case of a simple field model which is an effective low-energy description of *QCD*. Moreover, we can take advantage of the results in this model to quantitatively predict the precision of the results in the determination of parameters of resonances in *QCD*; we plan to deal with the glueball resonance.

## THEORETICAL BACKGROUND

In lattice field theory we determine the mass of a stable particle from the exponential decay of suitable correlation functions in Euclidean space. If  $G(x, y)$  is the two-point Green function of a generic field  $\phi$ , we can consider the

partial Fourier transform

$$C(t, \vec{p}) = \int \frac{d^3\vec{x}}{(2\pi)^3} e^{-i\vec{p}\vec{x}} G(\vec{x}, t; \vec{0}, 0) = \int \frac{d\omega}{2\pi} e^{i\omega t} \tilde{G}(\vec{p}, \omega), \quad (1)$$

where  $\tilde{G}(\vec{p}, \omega)$  is the Fourier transform of  $G(x, y)$ . One then expects that at large times  $t$

$$C(t, \vec{p} = 0) \propto e^{-m_\phi t}, \quad (2)$$

where  $m_\phi$  is the lightest mass which appears in the spectrum of the theory. On the other hand, if we consider an unstable particle  $\phi$  which can decay in two particles  $\pi$  ( $\phi \rightarrow 2\pi$ ) when  $m_\phi > 2m_\pi$ , at large times  $t$  we have

$$C(t, \vec{p} = 0) \propto e^{-2m_\pi t}, \quad (3)$$

and so we have no information about the mass of  $m_\phi$ .

Theoretically, we can characterize a resonance studying the  $S$ -matrix theory; let us consider the element  $S_l(E)$  ( $l$  is the angular momentum and  $E$  is the energy, considered as a complex variable) then we can define a resonance as the pole  $\tilde{E}$  of  $S_l(E)$ , in the complex energy plane, when  $\text{Im}(\tilde{E}) < 0$  and  $\text{Re}(\tilde{E}) \neq 0$ . In fact, if  $\text{Im}(\tilde{E})$  is small it is possible to show that the presence of the pole appears as a *peak* in the total cross section, for the corresponding  $l$ -wave contribution:

$$\sigma_l(E) \propto \frac{1}{E} \frac{\Gamma^2/4}{(E - M)^2 + \Gamma^2/4} \quad (4)$$

where  $M \propto \text{Re}(\tilde{E})$  and  $\Gamma \propto \text{Im}(\tilde{E})$ . As a consequence, a resonance is characterized not only by a mass  $M$  but also by a width  $\Gamma$ ; therefore, we cannot identify a resonance with an isolated energy level. This has to be compared with the fact that the energy spectrum of the Euclidean theory is always real, i.e. we can only study isolated levels. It is clear that it is not possible to get the parameters of a resonance using the standard tools of the lattice approach.

As explained in the introduction, if a resonance is present in a theory, it manifests itself as a rearrangement of the energy levels related to a mixing between the unstable particle  $\phi$  and the two stable particles  $\pi$ ; taking advantage of this effect, i.e. using the Lüscher approach, it is possible to obtain the resonance parameters.

## Interacting particles in a box

Let us consider a system of two identical non-interacting bosons of mass  $m_\pi$  in a box of volume  $V = \prod_{i=1}^3 L_i$ . The two-particle states are characterized by a total energy  $E$ , in the rest frame of the box, and by a relative momentum  $\vec{p}$  related by:

$$E = 2\sqrt{m_\pi^2 + |\vec{p}|^2}. \quad (5)$$

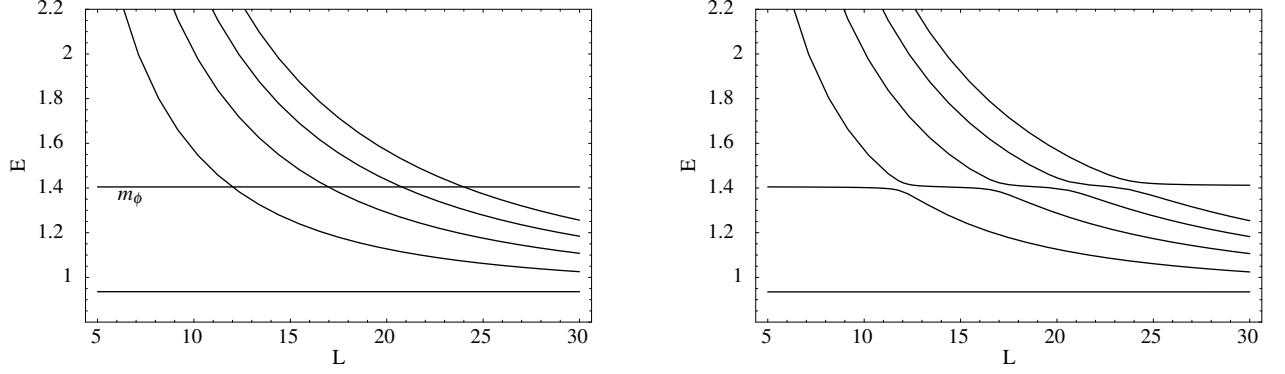
The quantized momentum  $p_i$  are given by  $p_i = \frac{2\pi}{L_i} n_i$ , where  $n_i \in \mathbb{Z}$ . Note that Eq. (5) is only an approximation on the lattice because it does not take in account space-time discretization artefacts.

We can study the spectrum of this system using Eq. (5); in particular, we can determine it for different values of  $L_i$  and for different values of  $\vec{n}$  ( $\vec{n} = (n_1, n_2, n_3)$ ). In a cubic box, if  $n^2 = \sum_{i=1}^3 n_i^2$  is fixed, we have degenerate energy levels for different values of  $n_i$ . In Figure 1 (Left) we show the five lowest levels as function of  $L$  (cubic box). As proposed in [16] can be an advantage to study the scattering using an asymmetric box; in fact, this allows to access more non-degenerate low-momentum modes for a given volume.

Because we are interested in resonant scattering, let us introduce a new particle in the system with mass  $m_\phi$  related to the physical mass of  $\pi$  by  $2m_\pi < m_\phi < 4m_\pi$ ; at the moment, it is not interacting with the particles  $\pi$ . In Figure 1 (Left) the  $\phi$  energy level is the horizontal line; it intersects the two-particle levels at various system sizes  $L$ . If we introduce a three point interaction between  $\phi$  and  $\pi$ , in a Minkowski space and in an infinite volume, we can say the particle  $\phi$  becomes unstable and it decays into two  $\pi$ -particles. In Euclidean space and in a finite volume we can only say that, because of the interaction, the energy eigenstates are a mixture of  $\phi$  and  $2\pi$ . This is manifested as ALCs in the energy levels as shown in Figure 1 (Right).

When the interaction is turned on Eq. (5) is still valid, but the momentum quantization condition is replaced with the Lüscher formula which connects the momentum  $p = |\vec{p}|$  with the scattering phase shift  $\delta_l$ :

$$\delta_l(p) = -\phi(q) \bmod \pi, \quad q = \frac{pL}{2\pi}, \quad (6)$$



**FIGURE 1.** (Left) The spectrum of a system of two non-interacting particles of mass  $m_\pi = 0.468$ ; the horizontal line describes the particle  $\phi$  at rest of mass  $m_\phi = 1.405$ . With these parameters the intersection between  $\phi$  and the two-particle level  $n^2 = 1$  is set at  $L = 12$ . (Right) Avoided level crossings where on the Left there were intersections between  $\phi$  and  $2\pi$ .

where  $\phi(q)$  is determined when  $l = 0$  by

$$\tan \phi(q) = -\frac{\pi^{3/2}q}{Z_{00}(1;q^2)}, \quad \phi(0) = 0. \quad (7)$$

The function  $Z_{00}(s;q^2)$  is the generalized zeta function defined in [3]. The determination of the resonance parameters follows immediately. Using Monte Carlo simulations we can determine the two-particle energy spectrum (a plot looking like Figure 1 (Right)). Then employing Eq. (5) for each value of  $E(L)$  we can determine the corresponding value of  $p(L)$  and from Eq. (6) the value of  $\delta(E)$ . Near the resonance we can fit  $\delta(E)$  to the relativistic Breit-Wigner formula

$$\tan \left( \delta_l - \frac{\pi}{2} \right) = \frac{E^2 - m_\phi^2}{m_\phi \Gamma_\phi}, \quad (8)$$

and therefore we can determine the two parameters  $m_\phi$  and  $\Gamma_\phi$ . Unfortunately, the method described up to now gives poor results when the width of the resonance is large (see the  $\Delta$  resonance case, analyzed in [17]). In this case the ALCs are washed out, i.e. we cannot locate the presence of the resonance in the way possible using data like that shown in Figure 1 (Right).

## Probability distribution method

An alternative method to solve the previous problem is based on a different way to *analyze* the finite volume energy spectrum [15]. The basic idea is to construct the probability distribution  $W(p)$  according to the prescriptions:

1. Measure the two-particle spectrum  $E_n(L)$  for different values of  $L$  and for  $n = 1, \dots, N$ ;
2. Determine  $p_n(L)$  using Eq. (5), for each  $n$  and  $L$ ;
3. Choose a suitable momentum interval  $[p_1, p_2]$  and introduce an equal-size momentum bin with length  $\Delta p$ ;
4. Count how many eigenvalues  $p_n(L)$  are contained in each bin;
5. Normalize this distribution in the interval  $[p_1, p_2]$ .

It is possible to show that the probability distribution  $W(p)$  is given by  $W(p) = c \sum_{n=1}^N [p'_n(L)]^{-1}$  and differentiating Eq. (6) with respect to  $L$ , it turns out:

$$W(p) = \frac{c}{p} \sum_{n=1}^N \left[ L_n(p) + \frac{2\pi\delta'(p)}{\phi'(q_n(p))} \right], \quad [c \text{ is a normalization constant}]. \quad (9)$$

Let us introduce  $W_0(p)$ , determined by Eq. (9) with  $\delta(p) = 0$  and  $L_n(p)$  corresponding to the free energy levels. The authors of [15] showed that in order to subtract the background (free  $\pi$  particles) it is convenient to consider the

subtracted probability distribution  $\tilde{W}(p) = W(p) - W_0(p)$ . In the infinite volume limit, this last quantity is determined by  $\delta(p)$  alone and close to the resonance, assuming a smooth dependence on  $p$  for the other quantities, it follows a Breit-Wigner shape with the *same width*. The main task of our work is to test this method on an effective field theory where a resonance emerges and then exploit the qualitative and quantitative knowledge in the real case of *QCD*.

## A SIMPLE MODEL OF SCALAR MESON DECAY

We examine a non-linear sigma model, characterized by a field  $\Sigma \in SU(2)$ , coupled to a scalar field  $\phi$  by a three-point interaction. The action is given by

$$S = \int d^4x \left\{ \beta \text{Tr} (\partial_\mu \Sigma_x \partial_\mu \Sigma_x^\dagger) - m_\pi^2 \beta \text{Tr} (\Sigma_x + \Sigma_x^\dagger) + \frac{1}{2} \phi_x (-\partial_\mu \partial_\mu + m_\phi^2) \phi_x + \lambda \phi_x \beta \text{Tr} (\partial_\mu \Sigma_x \partial_\mu \Sigma_x^\dagger + v) \right\}, \quad (10)$$

where the relation between the field  $\Sigma$  and the three (pion) fields  $\pi_i$  is given by  $\Sigma_x = \exp \left( i \frac{\pi_i \sigma_i}{f} \right)$ . The coupling  $\beta$  is related to the pion decay constant  $f$  by  $\beta = f^2/4$ . The coefficient  $v$  is introduced and tuned so that  $\langle \phi \rangle = 0$ . The action is invariant under global chiral transformation  $\Sigma \rightarrow U_R \Sigma U_L^\dagger$ , where  $U_L, U_R \in SU(2)$ , apart from an explicit symmetry breaking term  $m_\pi^2 \beta \text{Tr} (\Sigma_x + \Sigma_x^\dagger)$ . It is instructive to consider the action for  $\beta \rightarrow \infty$ , i.e. when the three pions do not interact with each other. In this limit,

$$S = \frac{1}{2} \int d^4x \left\{ \pi_x^i (-\partial_\mu \partial_\mu + m_\pi^2) \pi_x^i + \phi_x (-\partial_\mu \partial_\mu + m_\phi^2) \phi_x + \lambda \phi_x \partial_\mu \pi_x^i \partial_\mu \pi_x^i + \lambda f^2 v \phi_x \right\}, \quad (11)$$

and we can see clearly the characteristic coupling term where the derivatives of the two pions appear. When we discretize this theory on the lattice we note it is possible to update both fields  $\phi$  and  $\Sigma$  by means of the heatbath algorithm [18]. Naturally, this model closely resembles a linear sigma model, although the  $\phi\pi\pi$  interactions differ slightly. A “bottom up” construction was used to ensure similar prescriptions could be followed to couple mesons with general spin. An example of this for the  $\rho$ -meson is found in [5].

## MONTE CARLO SIMULATION

In order to determine the two-particle spectrum we first introduce the partial Fourier transform (PFT) of the field  $\pi$ :

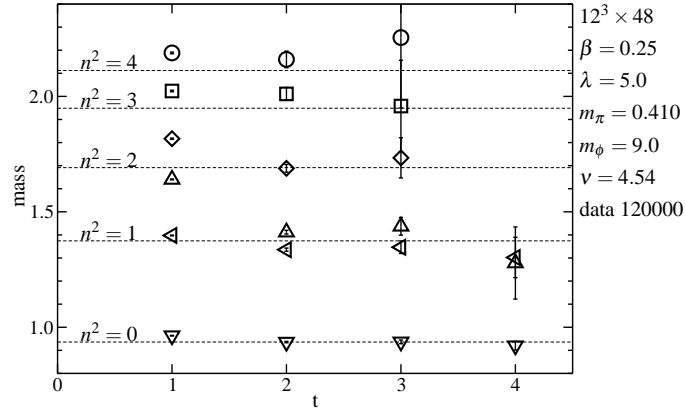
$$\tilde{\pi}(\vec{n}, t) = \frac{1}{V} \sum_x \pi(\vec{x}, t) e^{i\vec{x}\vec{p}}, \quad p_i = \frac{2\pi}{L_i} n_i, \quad n_i = 0, \dots, L_i - 1. \quad (12)$$

Then we study operators with zero total momentum and zero isospin:

$$O_{\vec{n}}(t) = \sum_{i=1}^3 \tilde{\pi}^i(\vec{n}, t) \tilde{\pi}^i(-\vec{n}, t); \quad (13)$$

in particular we take in account five different operators, corresponding to  $n^2 = 0, 1, 2, 3, 4$ . A sixth operator, that clearly has the correct quantum number, is the PFT of the field  $\phi$  with  $\vec{p} = 0$ . To determine the energy levels we use a method, introduced in [8], based on a generalized eigenvalue problem applied to the correlation matrix function  $C_{ij}(t) = \langle O_i O_j \rangle$ , i.e. a matrix whose elements are all possible correlators between the six operators. An example of spectrum is given in Figure 2 where the dashed lines represent the free two-pion spectrum. In this plot the parameters have been chosen to show an ALC between the  $\phi$  level and the  $2\pi$  level for  $n^2 = 1$ . The gap between these two levels is small compared to the difference between the others (the width of the resonance is small) and therefore it is simple to determine the position of the resonance, i.e. in this case the ALC is not washed out.

Unfortunately, it looks like it is not possible to tune the parameters of this model to increase the width of the resonance arbitrarily. Two different problems arise, depending on the value of  $\beta$ . If  $\beta < 0.25$ , a saturation effect appears so even if we increase the value of  $\lambda$ , the width of the resonance does not increase. On the other hand if  $\beta$  is big, a vacuum instability effect prevents simulations beyond a  $\lambda_{max}$  value. At the moment, we are trying to understand the origin of these problems in more detail through perturbative calculations.



**FIGURE 2.** The two-pion spectrum. The ALC is seen around the  $n^2 = 1$  two-pion state. The dashed lines, which show the free two-pion spectrum, take in account the discretization artefacts.

## CONCLUSION AND OUTLOOK

The problem of studying the width of a resonance on the lattice has attracted much attention recently. The main tool to get results is based on the study of the volume dependence of the two-particle states spectrum. A new method to analyze the data, based on the concept of probability distribution, has been proposed recently. In this work we have started to study a lattice field model where the new method will be tested. This model is important not only from a qualitative point of view, but it will be quantitatively important: we can tune the parameters of models like this to obtain correlation functions with the same precision of a QCD Monte Carlo study and therefore predict the precision with which we can obtain the QCD resonance parameters. Investigations of the scalar glueball following a prescription like this have begun.

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